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A new and elementary $\mathbb{C}P^n$ Dyonic magnon

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ABSTRACT: We show that the dressing transformation method produces a new type of dyonic $\mathbb{C}P^n$ magnon in terms of which all the other known solutions are either composites or arise as special limits. In particular, this includes the embedding of Dorey's dyonic magnon via $\mathbb{R}P^3 \subset \mathbb{C}P^n$. We also show how to generate Dorey's dyonic magnon directly in the S^n sigma model via the dressing method without resorting to the isomorphism with the $SU(2)$ principle chiral model when $n = 3$. The new dyon is shown to be either a charged dyon or topological kink of the related symmetric-space sine-Gordon theories associated to $\mathbb{C}P^n$ and in this sense is a direct generalization of the soliton of the complex sine-Gordon theory.

KEYWORDS: Solitons Monopoles and Instantons, AdS-CFT Correspondence, Integrable Field Theories

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1 Introduction

Motivated by the investigation of the AdS/CFT correspondence for $AdS_4 \times CP^3$ [1], the CP^n giant magnons have been recently discussed in some detail [2–9]. Similarly to their S^n counterparts, they are soliton solutions to the equations of motion of a Pohlmeyer reduced sigma model with target space CP^n .

So far, four different kinds of CP^n giant magnons have been described in the literature. Two of them are obtained by embedding the original S^2 Hofman-Maldacena giant magnon [10] in two distinct subspaces; namely, $CP^1 \subset CP^n$ [2], and $RP^2 \subset CP^n$ ($n \geq 2$) [3]. They have one parameter, and carry a single non-vanishing conserved charge (angular momentum). The third one is obtained by embedding Dorey’s dyonic S^3 giant magnon [11] via $RP^3 \subset CP^n$ ($n \geq 3$) [6], and it is a two-parameter (dyonic) generalization of the RP^2 magnon that carries two conserved charges. The fourth CP^n magnon was recently constructed by the present authors in [12] using the dressing method (see also [7, 8]). It has two parameters and takes values in a CP^2 subspace but carries only a single conserved charge. The relationship between all these (sigma model) giant magnons and those obtained from algebraic curves is discussed in [9].

The purpose of this note is to argue that all those magnons can be built out of a new type of CP^n dyonic magnon that we construct using the dressing transformation method. The existence of additional CP^n dyonic solutions was conjectured in [12] as a consequence of the general form of the metric on the moduli space of internal collective

coordinates. Moreover, the main features of a solution of precisely the type of the new dyon were discussed in [9], although its explicit form was only found there for a particular value of the parameters. Our conclusion will be that the dressing method can produce all known solutions, either as composites of the new dyonic magnon, or as special limits of a single one. In particular, we will argue that the embedding of Dorey’s dyonic magnon in a subspace $\mathbb{R}P^3 \subset \mathbb{C}P^n$ is a composite configuration of two of the new dyonic magnons and has internal moduli corresponding to separating the constituents. As a by-product we are able to show how Dorey’s magnon can be constructed in the S^n theory directly by using the dressing method. In this context, it is a fundamental object which cannot be “pulled apart”.

One of the main results of [12] was that the dressing method naturally produces magnon solutions in the original sigma model and at the same time the associated solution — the “solitonic avatar” — in the associated Symmetric-Space Sine-Gordon (SSSG) theory. This fact allows us to also investigate the nature of the avatar of the new dyonic magnon.

The paper is organized as follows. In section 2 we review the construction of the $\mathbb{C}P^n$ sigma model and in section 3 we describe how to impose the Pohlmeyer reduction and then how to construct solutions using the dressing transformation. In section 4 we construct the simplest kinds of solution and in particular previously over-looked solutions that are the new elementary dyonic magnons. In section 5 we consider the new solutions from the point-of-view of the associated SSSG system of equations. Finally in section 6 we show that all the known solutions can be recovered from the new dyons.

2 The $\mathbb{C}P^n$ sigma model

The $2n$ -dimensional complex projective space

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1}}{\mathbf{Z} \sim \lambda \mathbf{Z}} \simeq \frac{\text{SU}(n+1)}{\text{U}(n)}, \tag{2.1}$$

where \mathbf{Z} is a complex $n+1$ dimensional vector and $\lambda \in \mathbb{C}$, is a compact symmetric space F/G specified by the involution

$$\sigma_-(f) = \theta f \theta, \tag{2.2}$$

where

$$f \in \text{SU}(n+1), \quad \theta = \text{diag}(-1, 1, \dots, 1). \tag{2.3}$$

Acting on \mathfrak{f} , which is the Lie algebra of $F = \text{SU}(n+1)$, it gives rise to the canonical decomposition

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p} \quad \text{with} \quad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}, \quad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}, \tag{2.4}$$

where, using the fundamental representation of $\text{SU}(n+1)$, the form of the elements $r \in \mathfrak{g}$ and $k \in \mathfrak{p}$ is

$$r = \begin{pmatrix} in\phi & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & -i\phi I_{n \times n} + \mathcal{M} & \\ 0 & & & \end{pmatrix}, \quad k = \begin{pmatrix} 0 & v_1 & \dots & v_n \\ -v_1^* & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ -v_n^* & 0 & \dots & 0 \end{pmatrix}, \tag{2.5}$$

with \mathcal{M} a $n \times n$ anti-Hermitian matrix. Notice that \mathfrak{g} is the Lie algebra of $G = U(n)$.

Following the approach of [12, 13], the sigma model with target space $\mathbb{C}P^n$ can be formulated in terms of a $SU(n+1)$ -valued field \mathcal{F} subject to the constraint

$$\sigma_-(\mathcal{F}) = \theta \mathcal{F} \theta = \mathcal{F}^{-1}. \tag{2.6}$$

The map from the space $\mathbb{C}P^n$ into this field is given by

$$\mathcal{F} = \theta \left(I - 2 \frac{\mathbf{Z} \mathbf{Z}^\dagger}{|\mathbf{Z}|^2} \right), \tag{2.7}$$

where \mathbf{Z} is a complex $n+1$ dimensional vector whose components are the complex projective (embedding) coordinates. Then, the Lagrangian of the sigma model is

$$\mathcal{L} = -\text{Tr}(\mathcal{J}_\mu \mathcal{J}^\mu) \quad \text{with} \quad \mathcal{J}_\mu = \partial_\mu \mathcal{F} \mathcal{F}^{-1}, \tag{2.8}$$

whose equations-of-motion of (2.8) are

$$\partial_\mu \mathcal{J}^\mu = 0. \tag{2.9}$$

They exhibit that \mathcal{J}_μ is the conserved current corresponding to the global symmetry transformation¹

$$\mathcal{F} \rightarrow U \mathcal{F} \sigma_-(U^{-1}), \quad U \in SU(n+1), \tag{2.10}$$

which gives rise to the conserved Noether charge

$$\mathcal{Q}_L = \int_{-\infty}^{+\infty} \partial_0 \mathcal{F} \mathcal{F}^{-1}. \tag{2.11}$$

3 Pohlmeyer reduction and dressing transformations

“Giant magnon” is the name given to a soliton of the Pohlmeyer reduced sigma model in the context of string theory. For the $\mathbb{C}P^n$ sigma model, the Pohlmeyer reduction involves imposing the conditions [12, 14]²

$$\partial_\pm \mathcal{F} \mathcal{F}^{-1} = f_\pm \Lambda f_\pm^{-1}, \tag{3.1}$$

where $f_\pm \in SU(n+1)$ and

$$\Lambda = \left(\begin{array}{cc|c} 0 & -1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right). \tag{3.2}$$

Pohlmeyer reduction gives rise to an associated relativistic integrable system that is a generalization of the sine-Gordon theory [15]. These are the SSSG theories, and giant

¹The Lagrangian density (2.8) is invariant under the global transformations $\mathcal{F} \rightarrow U \mathcal{F} V$ for any $U, V \in SU(n+1)$. However, this symmetry is reduced by the constraint (2.6) so that the $\mathbb{C}P^n$ sigma model is invariant only under (2.10).

²In our notation, $x_\pm = t \pm x$ and $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$.

magnons can be mapped into the soliton solutions of their equations-of-motion. These latter equations are

$$\left[\partial_+ + \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+^{(L)} \gamma - \frac{1}{2} \Lambda, \partial_- + A_-^{(R)} - \frac{1}{2} \gamma^{-1} \Lambda \gamma \right] = 0, \quad (3.3)$$

where

$$\gamma = f_-^{-1} f_+ \quad (3.4)$$

is the SSSG group field, which takes values in $G \subset F$. The quantities $A_+^{(L)}$ and $A_-^{(R)}$ can be interpreted as components of gauge fields taking values in \mathfrak{h} , the Lie algebra of $H \subset G$, which is the subgroup of elements that commute with Λ . They are given by

$$\begin{aligned} A_+^{(L)} &= f_-^{-1} \partial_+ f_- - \frac{1}{2} \gamma \Lambda \gamma^{-1}, \\ A_-^{(R)} &= f_+^{-1} \partial_- f_+ - \frac{1}{2} \gamma^{-1} \Lambda \gamma. \end{aligned} \quad (3.5)$$

In the present case, $F = \text{SU}(n+1)$, $G = \text{U}(n)$ and $H = \text{U}(n-1)$.

We will be interested in the $\mathbb{C}P^n$ magnons constructed using the dressing transformation method [16, 17], which was shown to be consistent with the Pohlmeyer reduction in [12]. The procedure begins by identifying a ‘‘vacuum’’ solution which, in the present context, will be the simplest one which naturally satisfies the constraints (3.1) with $f_{\pm} = I$. It corresponds to

$$\mathcal{F}_0 = \left(\begin{array}{cc|c} \cos 2t & -\sin 2t & \mathbf{0} \\ +\sin 2t & \cos 2t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{array} \right), \quad \mathbf{Z}_0 = \mathbf{e}_1 \cos t - \mathbf{e}_2 \sin t, \quad (3.6)$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ is a set of real orthonormal vectors in \mathbb{C}^{n+1} and we have highlighted the 2×2 subspace associated to \mathbf{e}_1 and \mathbf{e}_2 .

The dressing transformation method makes use of the associated linear system

$$\partial_{\pm} \Psi(x; \lambda) = \frac{\partial_{\pm} \mathcal{F} \mathcal{F}^{-1}}{1 \pm \lambda} \Psi(x; \lambda), \quad \Psi(x; \infty) = I, \quad \mathcal{F}(x) = \Psi(x; 0), \quad (3.7)$$

whose integrability conditions are equivalent to the equations of motion of the sigma model. For $\mathbb{C}P^n$, the solutions $\Psi(x; \lambda)$ have to satisfy the two conditions

$$\Psi^{-1}(x; \lambda) = \Psi^{\dagger}(x; \lambda^*), \quad \Psi(x; 1/\lambda) = \mathcal{F} \theta \Psi(x; \lambda) \theta, \quad (3.8)$$

which ensure that $\mathcal{F}^{-1} = \mathcal{F}^{\dagger}$ and that the constraint (2.6) is satisfied. Then, the dressing transformation involves constructing a new solution Ψ of the linear system of the form

$$\Psi(x; \lambda) = \chi(x; \lambda) \Psi_0(x; \lambda) \quad (3.9)$$

in terms of an old one, which in our case correspond to the vacuum solution in (3.6):

$$\Psi_0(x; \lambda) = \exp \left[\left(\frac{x_+}{1+\lambda} + \frac{x_-}{1-\lambda} \right) \Lambda \right]. \quad (3.10)$$

Following [17], the general form of the “dressing factor” is

$$\chi(\lambda) = 1 + \sum_i \frac{Q_i}{\lambda - \lambda_i}, \quad \chi^{-1}(\lambda) = 1 + \sum_i \frac{R_i}{\lambda - \mu_i}, \quad (3.11)$$

where the residues are matrices of the form

$$Q_i = \mathbf{X}_i \mathbf{F}_i^\dagger, \quad R_i = \mathbf{H}_i \mathbf{K}_i^\dagger \quad (3.12)$$

for vectors \mathbf{X}_i , \mathbf{F}_i , \mathbf{H}_i , and \mathbf{K}_i . For $\mathbb{C}P^n$, they are given by

$$\mathbf{X}_i \Gamma_{ij} = \mathbf{H}_j, \quad \mathbf{K}_i (\Gamma^\dagger)_{ij} = -\mathbf{F}_j, \quad \Gamma_{ij} = \frac{\mathbf{F}_i^\dagger \mathbf{H}_j}{\lambda_i - \mu_j}, \quad (3.13)$$

$$\mathbf{F}_i = \Psi_0(\lambda_i^*) \boldsymbol{\varpi}_i, \quad \mathbf{H}_i = \Psi_0(\mu_i) \boldsymbol{\pi}_i,$$

where $\boldsymbol{\varpi}_i$ and $\boldsymbol{\pi}_i$ are complex constant $n + 1$ dimensional vectors. The allowed number of poles and their positions are constrained by the conditions (3.8). They imply that $\mu_i = \lambda_i^*$ and, moreover, that the poles $\{\lambda_i\}$ must come in pairs $(\lambda_i, \lambda_{i+1} = 1/\lambda_i)$. In addition, $\boldsymbol{\pi}_i = \boldsymbol{\varpi}_i$ and, for each pair,

$$\boldsymbol{\varpi}_{i+1} = \theta \boldsymbol{\varpi}_i. \quad (3.14)$$

In [12], it was shown that the value of the $SU(n+1)$ charge carried by these “dressed” solutions, relative to the vacuum solution, can be easily calculated in terms of the asymptotic values of the residues by means of

$$\Delta Q_L = \sum_i Q_i \Big|_{x=+\infty} - \sum_i Q_i \Big|_{x=-\infty}. \quad (3.15)$$

One the main results of [12] is that the dressing transformation not only produces the magnon solutions but the associated solitons, the “avatars”, of the related Symmetric Space Sine-Gordon (SSSG) equation. They are given by

$$\gamma = \mathcal{F}_0^{-1/2} \chi(+1)^{-1} \chi(-1) \mathcal{F}_0^{1/2}, \quad (3.16)$$

with $A_+^{(L)} = A_-^{(R)} = 0$. The group field also satisfies the constraints

$$\gamma^{-1} \partial_+ \gamma \Big|_{\mathfrak{h}} = \partial_- \gamma \gamma^{-1} \Big|_{\mathfrak{h}} = 0. \quad (3.17)$$

4 Magnons by dressing the vacuum

We shall consider in detail the soliton solutions obtained from a single pair of poles $\{\xi, 1/\xi\}$, with $\xi = r e^{ip/2}$. The dressing factor is [12]

$$\chi(\lambda) = 1 + \frac{Q_1}{\lambda - \xi} + \frac{Q_2}{\lambda - 1/\xi}, \quad (4.1)$$

where

$$Q_1 = \frac{1}{\Delta} \left[-\frac{|\xi^2| \beta}{\xi - \xi^*} \mathbf{F} \mathbf{F}^\dagger + \frac{\xi \gamma}{|\xi|^2 - 1} \mathcal{F}_0 \theta \mathbf{F} \mathbf{F}^\dagger \right], \quad (4.2)$$

$$Q_2 = \frac{1}{\Delta} \left[\frac{\beta}{\xi - \xi^*} \mathcal{F}_0 \theta \mathbf{F} \mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger - \frac{\xi^* \gamma}{|\xi|^2 - 1} \mathbf{F} \mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger \right],$$

and we have defined the real numbers

$$\beta = \mathbf{F}^\dagger \mathbf{F}, \quad \gamma = \mathbf{F}^\dagger \mathcal{F}_0 \theta \mathbf{F}, \quad \Delta = \frac{|\xi|^2 \gamma^2}{(|\xi|^2 - 1)^2} - \frac{|\xi|^2 \beta^2}{(\xi - \xi^*)^2}. \quad (4.3)$$

Here,

$$\mathbf{F} = \Psi_0(\xi^*) \varpi \quad (4.4)$$

where ϖ is a complex $n + 1$ dimensional vector. Then, the magnon solution is given by $\mathcal{F} = \Psi(0) = \chi(0)\mathcal{F}_0$, which corresponds to the projective coordinates [12] (see also [18])

$$\mathbf{Z} = (\tilde{\alpha} + \theta \mathbf{F} \mathbf{F}^\dagger \theta) \mathbf{Z}_0, \quad (4.5)$$

with

$$\tilde{\alpha} = -\frac{\xi \beta}{\xi - \xi^*} - \frac{\gamma}{|\xi|^2 - 1}. \quad (4.6)$$

In the following, it will be useful to introduce the notation

$$f(\xi^*) = -i \left(\frac{x_+}{1 + \xi^*} + \frac{x_-}{1 - \xi^*} \right) = 2F(t, x) - iG(t, x), \quad (4.7)$$

where

$$\begin{aligned} F(t, x) &= \frac{1}{2} x' \cos \alpha = \frac{(1 + r^2)r \sin \frac{p}{2}}{(1 - r^2)^2 + 4r^2 \sin^2 \frac{p}{2}} x - \frac{r^2 \sin p}{(1 - r^2)^2 + 4r^2 \sin^2 \frac{p}{2}} t, \\ G(t, x) &= t - t' \sin \alpha = -\frac{2(1 - r^2)r \cos \frac{p}{2}}{(1 - r^2)^2 + 4r^2 \sin^2 \frac{p}{2}} x + \frac{2(1 - r^2 \cos p)}{(1 - r^2)^2 + 4r^2 \sin^2 \frac{p}{2}} t. \end{aligned} \quad (4.8)$$

The rapidity ϑ and the parameter α are determined in terms of r and p by

$$\tanh \vartheta = \frac{2r}{1 + r^2} \cos \frac{p}{2}, \quad \cot \alpha = \frac{2r}{1 - r^2} \sin \frac{p}{2}, \quad (4.9)$$

and the Lorentz boosted coordinates t' and x' are

$$t' = t \cosh \vartheta - x \sinh \vartheta, \quad x' = x \cosh \vartheta - t \sinh \vartheta. \quad (4.10)$$

Looking at (4.2), it is easy to see that the solutions corresponding to ϖ and $\varpi \rightarrow \lambda \varpi$, $\lambda \in \mathbb{C}$, are equivalent solutions. The components of ϖ represent a set of collective coordinates for the magnons whose interpretation will be clarified in section 6. In particular, some of the components of ϖ fix the position of the centre of the magnons/solitons. Their localized nature arises because ξ has a imaginary part and thus $\Psi_0(\xi^*)$ has an exponential dependence on x . Since Λ is anti-hermitian, the relevant dependence is [12]

$$\exp \left[i \operatorname{Im} \left(\frac{x_+}{1 + \xi^*} + \frac{x_-}{1 - \xi^*} \right) \Lambda \right] = \exp \left[2i F(t, x) \Lambda \right], \quad (4.11)$$

and this leads to exponential fall-off of the energy/charge density away from the centre which is located at the solution of

$$F(t, x) = F_0, \quad (4.12)$$

where F_0 is a constant determined by the components of ϖ . In particular, this equation shows that the velocity of the magnon is

$$v = \frac{2r}{1+r^2} \cos \frac{p}{2} = \tanh \vartheta. \quad (4.13)$$

Moreover, since $\Psi_0(\xi^*)$ always appear in the combination $\mathbf{F} = \Psi_0(\xi^*)\varpi$, a constant shift of the solitons in space and time act on the collective coordinates via³

$$\varpi \longrightarrow \exp \left[\left(\frac{\delta x_+}{1+\xi^*} + \frac{\delta x_-}{1-\xi^*} \right) \Lambda \right] \varpi. \quad (4.14)$$

The solutions provided by (4.5) give rise to different species of magnons. In order to find them out, we shall investigate the value of the $SU(n+1)$ charge $\Delta \mathcal{Q}_L$ by means of (3.15). Using (4.7), we can write

$$\Psi_0(\xi^*) = \frac{1}{2} e^{f(\xi^*)} \left(\begin{array}{cc|c} 1 & -i & \mathbf{0} \\ +i & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right) + \left(\begin{array}{cc|c} 0 & 0 & \mathbf{0} \\ 0 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I} \end{array} \right) + \frac{1}{2} e^{-f(\xi^*)} \left(\begin{array}{cc|c} 1 & +i & \mathbf{0} \\ -i & 1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right), \quad (4.15)$$

whose asymptotic behaviour can be easily worked out by noticing that

$$\lim_{x \rightarrow \pm\sigma\infty} F(t, x) = \pm\infty \quad \text{with} \quad \sigma = \text{sign} \left(r \sin \frac{p}{2} \right). \quad (4.16)$$

Now, if we split the components of the complex $(n+1)$ -vector ϖ as

$$\varpi = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \mathbf{\Omega}, \quad \mathbf{\Omega} \cdot \mathbf{e}_1 = \mathbf{\Omega} \cdot \mathbf{e}_2 = 0, \quad (4.17)$$

it is easy to show that

$$\begin{aligned} \beta &= \mathbf{F}^\dagger \mathbf{F} = \frac{1}{2} e^{4F} |\omega_1 - i\omega_2|^2 + \mathbf{\Omega}^\dagger \mathbf{\Omega} + \frac{1}{2} e^{-4F} |\omega_1 + i\omega_2|^2 \\ \gamma &= \mathbf{F}^\dagger \mathcal{F}_0 \theta \mathbf{F} = (|\omega_2|^2 - |\omega_1|^2) \cos(2(G-t)) \\ &\quad + (\omega_1 \omega_2^* + \omega_1^* \omega_2) \sin(2(G-t)) + \mathbf{\Omega}^\dagger \mathbf{\Omega}. \end{aligned} \quad (4.18)$$

Therefore, we can distinguish three cases.

4.1 $|\omega_1 \pm i\omega_2| \neq 0$

In this case,

$$\begin{aligned} \beta &\xrightarrow{x \rightarrow \pm\sigma\infty} \frac{1}{2} e^{\pm 4F} |\omega_1 \mp i\omega_2|^2 + \dots, \quad \gamma \xrightarrow{x \rightarrow \pm\sigma\infty} \text{finite} \\ \implies \Delta &\xrightarrow{x \rightarrow \pm\sigma\infty} -\frac{|\xi|^2}{(\xi - \xi^*)^2} \frac{1}{4} e^{\pm 8F} |\omega_1 \mp i\omega_2|^2 + \dots, \end{aligned} \quad (4.19)$$

³For the magnon solutions $\mathcal{F} = \chi(0)\mathcal{F}_0$, this transformation gives rise to a constant shift of the dressing factor $\chi(0)$ in space and time, but not of \mathcal{F}_0 which depends only on t . In contrast, it is completely equivalent to a constant shift of the solitonic avatars given by (3.16) in space and time.

and

$$\begin{aligned}
 \mathbf{F}\mathbf{F}^\dagger &\xrightarrow{x \rightarrow \pm\sigma\infty} \frac{1}{4} e^{\pm 4F} |\omega_1 \mp i\omega_2|^2 \begin{pmatrix} 1 & \mp i & \mathbf{0} \\ \pm i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \dots \\
 \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger &\xrightarrow{x \rightarrow \pm\sigma\infty} \frac{1}{4} e^{\pm 4F} |\omega_1 \mp i\omega_2|^2 \begin{pmatrix} 1 & \pm i & \mathbf{0} \\ \mp i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \dots .
 \end{aligned} \tag{4.20}$$

Therefore,

$$Q_1 \xrightarrow{x \rightarrow \pm\sigma\infty} \frac{1}{2} (\xi - \xi^*) \begin{pmatrix} 1 & \mp i & \mathbf{0} \\ \pm i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad Q_2 \xrightarrow{x \rightarrow \pm\sigma\infty} -\frac{1}{2} \frac{\xi - \xi^*}{|\xi|^2} \begin{pmatrix} 1 & \pm i & \mathbf{0} \\ \mp i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{4.21}$$

which, using (3.15), give rise to the charge

$$\Delta Q_L = \left(Q_1 + Q_2 \right) \Big|_{-\infty}^{+\infty} = \sigma \left[i(\xi - \xi^*)\Lambda + i \frac{\xi - \xi^*}{|\xi|^2} \Lambda \right] = -2 \frac{r^2 + 1}{|r|} \left| \sin \frac{p}{2} \right| \Lambda. \tag{4.22}$$

This case produces the magnon solution constructed in [12] (see also [7, 8]). Notice that the transformation (4.14), which corresponds to a constant shift in space and time, is equivalent to

$$\omega_1 \pm i\omega_2 \longrightarrow e^{\mp \delta f} (\omega_1 \pm i\omega_2) \tag{4.23}$$

where (see (4.7))

$$\delta f = -i \left(\frac{\delta x_+}{1 + \xi^*} + \frac{\delta x_-}{1 - \xi^*} \right) = 2\delta F - i\delta G. \tag{4.24}$$

Then,

$$\left| \frac{\omega_1 - i\omega_2}{\omega_1 + i\omega_2} \right| \longrightarrow e^{4\delta F} \left| \frac{\omega_1 - i\omega_2}{\omega_1 + i\omega_2} \right| \tag{4.25}$$

and, as explained in the paragraph around (4.12), the centre of this magnon is located at the solution of

$$F(t, x) = \frac{1}{4} \log \left| \frac{\omega_1 - i\omega_2}{\omega_1 + i\omega_2} \right|, \tag{4.26}$$

which clarifies the meaning of the collective coordinates ω_1 and ω_2 . Then, up to a shift of the soliton in space and time, we can always set $\omega_2 = 0$ and, using the invariance under complex re-scalings $\varpi \rightarrow \lambda\varpi$, we can also set $\omega_1 = i$, so that

$$\varpi = i\mathbf{e}_1 + \mathbf{\Omega}, \tag{4.27}$$

which is precisely the normalization used in [12].

4.2 $|\omega_1 + i\omega_2| = 0$.

In this case, $\omega_2 = +i\omega_1$ and thus

$$\begin{aligned} \beta &= 2|\omega_1|^2 e^{+4F} + \mathbf{\Omega}^\dagger \mathbf{\Omega}, & \gamma &= \mathbf{\Omega}^\dagger \mathbf{\Omega} \\ \implies \Delta \xrightarrow{x \rightarrow +\sigma\infty} & -\frac{|\xi|^2}{(\xi - \xi^*)^2} 4e^{+8F} + \dots \\ \Delta \xrightarrow{x \rightarrow -\sigma\infty} & \left[\frac{|\xi|^2}{(|\xi|^2 - 1)^2} - \frac{|\xi|^2}{(\xi - \xi^*)^2} \right] (\mathbf{\Omega}^\dagger \mathbf{\Omega})^2 + \dots, \end{aligned} \quad (4.28)$$

together with

$$\begin{aligned} \mathbf{F}\mathbf{F}^\dagger \xrightarrow{x \rightarrow +\sigma\infty} & |\omega_1|^2 e^{+4F} \left(\begin{array}{c|cc} 1 & -i & \mathbf{0} \\ +i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right) + \dots \\ \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger \xrightarrow{x \rightarrow +\sigma\infty} & |\omega_1|^2 e^{+4F} \left(\begin{array}{c|cc} 1 & +i & \mathbf{0} \\ -i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right) + \dots \\ \mathbf{F}\mathbf{F}^\dagger, \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger, \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger, \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger \xrightarrow{x \rightarrow -\sigma\infty} & \left(\begin{array}{c|cc} \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{\Omega}\mathbf{\Omega}^\dagger & \end{array} \right) + \dots \end{aligned} \quad (4.29)$$

This leads to

$$\begin{aligned} Q_1 \xrightarrow{x \rightarrow +\sigma\infty} & \frac{1}{2}(\xi - \xi^*) \left(\begin{array}{c|cc} 1 & -i & \mathbf{0} \\ +i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right), & Q_2 \xrightarrow{x \rightarrow +\sigma\infty} & -\frac{1}{2} \frac{\xi - \xi^*}{|\xi|^2} \left(\begin{array}{c|cc} 1 & +i & \mathbf{0} \\ -i & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right), \\ Q_1 + Q_2 \xrightarrow{x \rightarrow -\sigma\infty} & \frac{(\xi - \xi^*)(|\xi|^2 - 1)}{|\xi|^2} \left(\begin{array}{c|cc} \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{\Omega}\mathbf{\Omega}^\dagger / \mathbf{\Omega}^\dagger \mathbf{\Omega} & \end{array} \right) \end{aligned} \quad (4.30)$$

and, using (3.15), to the charge

$$\Delta \mathcal{Q}_L = (Q_1 + Q_2) \Big|_{-\infty}^{+\infty} = -\frac{1+r^2}{|r|} \left| \sin \frac{p}{2} \right| \Lambda - \frac{1-r^2}{|r|} \left| \sin \frac{p}{2} \right| \left(\begin{array}{c|cc} i\mathbf{1} & \mathbf{0} & \\ \mathbf{0} & -2i\mathbf{\Omega}\mathbf{\Omega}^\dagger / \mathbf{\Omega}^\dagger \mathbf{\Omega} & \end{array} \right). \quad (4.31)$$

It corresponds to a new dyonic solution specified by the projective coordinates

$$\begin{aligned} \mathbf{Z} &= \left[-\left(2|\omega_1|^2 e^{4F} + \mathbf{\Omega}^\dagger \mathbf{\Omega} \right) \frac{\xi}{\xi - \xi^*} - \frac{\mathbf{\Omega}^\dagger \mathbf{\Omega}}{|\xi|^2 - 1} \right] (\cos t \mathbf{e}_1 - \sin t \mathbf{e}_2) \\ &+ |\omega_1|^2 e^{4F} e^{-it} (\mathbf{e}_1 - i\mathbf{e}_2) - \omega_1^* e^{2F} e^{i(G-t)} \mathbf{\Omega}, \end{aligned} \quad (4.32)$$

which is apparently singular when $|\xi| = 1$. However, a regular solution in this limit can be constructed by imposing the additional condition that

$$\mathbf{\Omega}^\dagger \mathbf{\Omega} = 0 \implies \mathbf{\Omega} = 0. \quad (4.33)$$

This particular limit produces precisely the embedding of the Hofman-Maldacena S^2 magnon in $\mathbb{C}P^1$ considered in [2]. In fact, (4.32) is the dyonic generalization of the $\mathbb{C}P^1$ magnon whose existence was conjectured in [9] where only the explicit form of a solution with $p = \pi$ was provided. In appendix A, we show that the solution reported in [9] is recovered for the particular choice of parameters

$$p = \pi, \quad \Omega^\dagger \Omega = 2 \frac{r^2 - 1}{r^2 + 1} |\omega_1|^2. \quad (4.34)$$

In this case, the transformation (4.14), which corresponds to a constant shift in space and time, is equivalent to

$$\omega_1 \longrightarrow e^{+\delta f} \omega_1 \Rightarrow |\omega_1| \longrightarrow e^{2\delta F} |\omega_1| \quad (4.35)$$

where δf is defined in (4.24). Then, according to the discussion around (4.12), and taking into account that the solution depends on ω_1 and Ω only up to complex re-scalings $\omega_1 \rightarrow \lambda \omega_1$ and $\Omega \rightarrow \lambda \Omega$, the centre of this magnon is located at the solution of

$$F(t, x) = \frac{1}{2} \log \frac{|\omega_1|}{|\Omega|}. \quad (4.36)$$

Therefore, up to a shift of the soliton in space and time, we can always set $|\Omega| = |\omega_1|$ and, using the invariance under complex re-scalings $\varpi \rightarrow \lambda \varpi$, we can also set $\omega_1 = 1$, so that

$$\varpi = e_1 + i e_2 + \Omega, \quad (4.37)$$

with $|\Omega| = 1$.

Eq. (4.32) manifests the fact that the new solution takes values in the $\mathbb{C}P^2 \subset \mathbb{C}P^n$ subspace picked out by the three mutually orthogonal vectors $\{e_1, e_2, \Omega\}$. According to (4.31), its charge has two independent components. Namely,

$$\Delta Q_L = J_\Lambda \Lambda + J_h h_\Omega, \quad J_\Lambda = -\frac{1+r^2}{|r|} \left| \sin \frac{p}{2} \right|, \quad J_h = -\frac{1-r^2}{|r|} \left| \sin \frac{p}{2} \right|, \quad (4.38)$$

where

$$h_\Omega = \left(\begin{array}{c|c} i\mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & -2i\Omega\Omega^\dagger/\Omega^\dagger\Omega \end{array} \right) \quad (4.39)$$

is one of the generators of $H = U(n-1)$, which is the subgroup of elements of $G = U(n)$ that commute with Λ . These charges satisfy the relation

$$-J_\Lambda = \sqrt{J_h^2 + 4 \sin^2 \frac{p}{2}}. \quad (4.40)$$

In the AdS/CFT context [10, 11], the components J_Λ and J_h are identified, up to scaling, with $\Delta - \frac{1}{2}J$ and Q , respectively, where Δ is the scaling dimension of the associated operator in the CFT, and J and Q are two conserved $U(1)$ R -charges:⁴

$$\Delta - \frac{1}{2}J = -\sqrt{\frac{\lambda}{2}} J_\Lambda, \quad \frac{1}{2}Q = \sqrt{\frac{\lambda}{2}} J_h, \quad (4.41)$$

⁴To be specific, we use the same normalization as [9].

where λ is the 't Hooft coupling. Then, (4.40) becomes the celebrated dispersion relation

$$\Delta - \frac{1}{2}J = \sqrt{\frac{1}{4}Q^2 + 2\lambda \sin^2 \frac{p}{2}}. \quad (4.42)$$

Eq. (4.38) shows that all the inequivalent magnons of this type are obtained with $r > 0$ and $\sin \frac{p}{2} > 0$, and that the transformation $r \rightarrow 1/r$ corresponds to $J_h \rightarrow -J_h$.

4.3 $|\omega_1 - i\omega_2| = 0$

This is equivalent to $\omega_2 = -i\omega_1$. Again, using the invariance under complex re-scalings $\varpi \rightarrow \lambda\varpi$, we can set $\omega_1 = 1$, so that

$$\varpi = e_1 - ie_2 + \Omega. \quad (4.43)$$

However, this case does not give rise to new magnon solutions. Let us write

$$\widehat{\varpi} = \begin{pmatrix} \omega_1 \\ -i\omega_1 \\ \Omega \end{pmatrix} = -\theta \begin{pmatrix} \omega_1 \\ +i\omega_1 \\ -\Omega \end{pmatrix} \equiv -\theta\varpi, \quad (4.44)$$

where ϖ produces (4.32) with $\Omega \rightarrow -\Omega$. Then, using (3.8),

$$F[\widehat{\varpi}, \xi] = \Psi_0(\xi^*)\widehat{\varpi} = -\theta\mathcal{F}_0^{-1}\Psi_0(1/\xi^*)\varpi = -\theta\mathcal{F}_0^{-1}F[\varpi, 1/\xi], \quad (4.45)$$

where we have explicitly indicated the dependence of F on ϖ and ξ . This implies

$$\beta[\widehat{\varpi}, \xi] = \beta[\varpi, 1/\xi], \quad \gamma[\widehat{\varpi}, \xi] = \gamma[\varpi, 1/\xi], \quad \Delta[\widehat{\varpi}, \xi] = \Delta[\varpi, 1/\xi] \quad (4.46)$$

and, finally,

$$Q_1[\widehat{\varpi}, \xi] = Q_2[\varpi, 1/\xi]. \quad (4.47)$$

Therefore, the dressing factor(4.1) satisfies

$$\chi[\lambda; \widehat{\varpi}, \xi] = \chi[\lambda; \varpi, 1/\xi], \quad (4.48)$$

which shows that $\widehat{\varpi} = -\theta\varpi$ and ξ gives rise to the same solution as ϖ and $1/\xi$, and confirms that taking $\omega_2 = -i\omega_1$ also leads to solutions of the form (4.32). It is worth noticing that $f(1/\xi^*) = -f(\xi^*) - 2it$, which means that

$$\xi \rightarrow 1/\xi \Rightarrow F(t, x) \longrightarrow -F(t, x), \quad G(t, x) \longrightarrow -G(t, x) + 2t. \quad (4.49)$$

5 The solitonic avatars

In the present case, it is sufficient to consider the case of $\mathbb{C}P^2$ because the solutions in this case can then be simply embedded in $\mathbb{C}P^n$, $n > 2$, in the obvious way. Introducing the parameterization in [12]

$$\gamma = e^{a_L h} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta e^{i\varphi} & \sin \theta \\ 0 & -\sin \theta & \cos \theta e^{-i\varphi} \end{pmatrix} e^{-a_R h}, \quad (5.1)$$

where $h = i \text{diag}(1, 1, -2)$ is the generator of \mathfrak{h} , the Lie algebra of $H = \text{U}(n-1)$, the dyon solution (3.16) in the rest frame ($p = \pi$) is⁵

$$\begin{aligned} \tilde{\psi} &= \frac{4}{3} \frac{1-r^2}{r^2+1} t, \\ \psi &= -4 \tan^{-1} \left(\frac{1}{r} \tanh \frac{4rx}{r^2+1} \right), \\ \varphi &= -\frac{2}{3} \tan^{-1} \left(\frac{1}{r} \right) - \tan^{-1} \left(\frac{1-3r^2}{r^3-3r} \tanh \frac{4rx}{r^2+1} \right) + \tan^{-1} \left(\frac{1}{r} \tanh \frac{4rx}{r^2+1} \right), \\ \sin^2 \theta &= \frac{16r^2(r^2-1)}{(r^2+1)^2 \left(2r^2 + (r^2+1) \sinh^2 \frac{4rx}{r^2+1} \right)}, \end{aligned} \quad (5.2)$$

where

$$\psi = 2(a_L - a_R), \quad \tilde{\psi} = 2(a_L + a_R). \quad (5.3)$$

The field $\psi(x, t)$ has a kink-like behaviour with

$$\Delta\psi = \psi(\infty, t) - \psi(-\infty, t) = -8 \tan^{-1}(r^{-1}) = -2\pi - 4\alpha, \quad (5.4)$$

where α is defined by (4.9) for $p = \pi$ (vanishing rapidity), so $\cot \alpha = 2r/(1-r^2)$. Notice that the S^1 -valued field $\tilde{\psi}(x, t)$ has constant angular velocity $\frac{4}{3}(1-r^2)/(r^2+1) = \frac{4}{3} \sin \alpha$. In this sense the solution is metaphorically the dyon solution of four-dimensional gauge theories where the angular variable $\tilde{\psi}$ is the $\text{U}(1)$ charge angle of the magnetic monopole.

As described in [12, 14], the SSSG system can be written as the equations-of-motion of a Lagrangian field theory. For the $\mathbb{C}P^2$ theory there are two inequivalent ways of doing this corresponding to “vector gauging” and “axial gauging” which are related by a target-space T-duality symmetry [19]. (For $\mathbb{C}P^n$, $n > 2$, axial gauging is not possible.) For vector gauging, the field $\tilde{\psi}(x, t)$ is a gauge degree-of-freedom and is consequently “gauged away” leaving a theory with physical fields (φ, ψ, θ) and a Lagrangian density

$$\mathcal{L} = \partial_\mu \theta \partial^\mu \theta + \frac{1}{4} \partial_\mu \psi \partial^\mu \psi + \cot^2 \theta \partial_\mu (\psi + \varphi) \partial^\mu (\psi + \varphi) + 2 \cos \theta \cos \varphi. \quad (5.5)$$

The $H = \text{U}(1)$ symmetry $\psi \rightarrow \psi + a$ is broken by the vacuum configuration $\theta = 0$, $\varphi = 0$ and ψ arbitrary, and consequently the soliton (5.2) carries topological, or kink, charge

$$Q^T = \frac{1}{4} \Delta\psi h = \left(-\frac{\pi}{2} - \alpha \right) h. \quad (5.6)$$

On the contrary, for the axial-gauged theory, the field $\psi(x, t)$ is a gauge degree-of-freedom that is gauged away leaving a theory with physical fields $(\varphi, \tilde{\psi}, \theta)$ and a Lagrangian density

$$\begin{aligned} \mathcal{L} = \partial_\mu \theta \partial^\mu \theta + \frac{1}{1+4 \cot^2 \theta} \left(\frac{9}{4} \partial_\mu \tilde{\psi} \partial^\mu \tilde{\psi} + \cot^2 \theta \partial_\mu \varphi \partial^\mu \varphi \right. \\ \left. - 6 \cot^2 \theta \epsilon^{\mu\nu} \partial_\mu \tilde{\psi} \partial_\nu \varphi \right) + 2 \cos \theta \cos \varphi, \end{aligned} \quad (5.7)$$

⁵In the following, we have used the symmetries to shift x , ψ and $\tilde{\psi}$ by appropriate constants.

In the resulting theory, the $H = U(1)$ group survives as a genuine symmetry $\tilde{\psi} \rightarrow \tilde{\psi} + a$ (corresponding to vector transformations) with an associated Noether current. The vacuum is invariant under the symmetry and remains unbroken. The quantity $\psi(x, t)$ still plays a rôle since it determines the associated Noether current

$$J^\mu = \epsilon^{\mu\nu} (\mathcal{A}_\nu + \frac{1}{4} \partial_\nu \psi h), \tag{5.8}$$

In this case the gauge field vanishes and so, as expected, the Noether charge is the same as the topological charge of the vector-gauged theory:

$$Q^N = \int dx J_0 = \frac{1}{4} \Delta \psi h = \left(-\frac{\pi}{2} - \alpha \right) h. \tag{5.9}$$

The masses of the dyon follows from the additivity of the mass of the — now realized to be composite (see section 6)—soliton calculated in [12], giving

$$M_{\text{dyon}} = \frac{4r}{r^2 + 1} = 2 \cos \alpha. \tag{5.10}$$

It is interesting to note that the dyon is a generalization of the soliton (or dyon) of the complex sine-Gordon theory [20]. It would be interesting to follow the approach of [20] to quantize the $\mathbb{C}P^2$ model (see also [21]).

6 Constructing all the known $\mathbb{C}P^n$ magnons

Since these new objects carry general charges we refer to them as “ $\mathbb{C}P^n$ dyons”. They are labelled by the data $(r, p, \mathbf{\Omega})$, corresponding to dressing data $\varpi = e_1 + ie_2 + \mathbf{\Omega}$ and $\xi = re^{ip/2}$, which determines their charges and rapidity (the latter via (4.9)). The overall magnitude and phase of $\mathbf{\Omega}$ determines the position and $U(1)$ “charge angle” of the dyon, respectively, leaving the equivalence class $\mathbb{C}P^{n-2} = \{\mathbf{\Omega} \sim \lambda \mathbf{\Omega}, \lambda \in \mathbb{C} \neq 0\}$ to specify its non-abelian orientation.⁶ These new dyonic magnons are the most elementary type of solution because all the other known solutions are either composites of them, or are obtained by taking special limits, as we now explain seriatim:

- (i) The original $\mathbb{C}P^n$ magnon of [12] corresponds to a configuration of two of the new dyons of the form

$$\left(r, p, \frac{1}{2} \mathbf{\Omega} \right) + \left(1/r, -p, -\frac{1}{2} \mathbf{\Omega} \right). \tag{6.1}$$

In order to see this, an inspection of eqs. (4.22) and (4.31) illustrates precisely that the magnon solution originally constructed in [12] (and in section 4.1) is to be thought

⁶In more detail we can think of $\mathbf{\Omega}$ as a real $2n - 2$ vector. The moduli space $\mathbb{R}^{2n-2} \simeq \mathbb{R}^+ \times S^{2n-3}$, for which the radius factor determines the dyon’s position. The S^{2n-3} can be viewed as a Hopf fibration of S^1 over a $\mathbb{C}P^{n-2}$ and the S^1 angle is associated to the $U(1)$ factor of the group $H = U(n - 1)$ (the centralizer of Λ in $G = U(n)$), this is the “charge angle” of the dyon which is rotating with constant angular velocity, while the $\mathbb{C}P^{n-2} \simeq SU(n - 1)/U(n - 2)$ factor describes the orientation of the dyon in the non-abelian subgroup $SU(n - 1) \subset H$.

of as a composite of two new dyonic magnons. In order to make this explicit, consider (4.15), which leads to

$$\mathbf{F} = \Psi_0(\xi^*)\boldsymbol{\varpi} = \frac{\omega_1 - i\omega_2}{2} e^{f(\xi^*)} \begin{pmatrix} 1 \\ +i \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\Omega} \end{pmatrix} + \frac{\omega_1 + i\omega_2}{2} e^{-f(\xi^*)} \begin{pmatrix} 1 \\ -i \\ \mathbf{0} \end{pmatrix}. \quad (6.2)$$

Then, since the solutions are constructed in terms of \mathbf{F} , we can indeed interpret this one as a superposition of two more basic constituents which are simply two dyonic magnons like those constructed in 4.2 associated to $(\xi, \frac{1}{2}\boldsymbol{\Omega})$ and $(1/\xi, -\frac{1}{2}\boldsymbol{\Omega})$. Using (4.36), they are mutually at rest at space-time positions determined by⁷

$$F_1(t, x_1) = \frac{1}{2} \log \left(\frac{|\omega_1 - i\omega_2|}{|\boldsymbol{\Omega}|} \right), \quad -F_2(t, x_2) = \frac{1}{2} \log \left(\frac{|\omega_1 + i\omega_2|}{|\boldsymbol{\Omega}|} \right). \quad (6.3)$$

Then, the centre of the composite soliton is at

$$F(t, x) = \frac{1}{2}(F_1(t, x_1) + F_2(t, x_2)) = \frac{1}{4} \log \left| \frac{\omega_1 - i\omega_2}{\omega_1 + i\omega_2} \right|, \quad (6.4)$$

in agreement with (4.26). Moreover, the distance between the two constituents is

$$\Delta x = x_1 - x_2 = \sec \alpha \operatorname{sech} \vartheta \log \left(\frac{|\omega_1^2 + \omega_2^2|}{\boldsymbol{\Omega}^\dagger \boldsymbol{\Omega}} \right). \quad (6.5)$$

The charges of the two constituents in the direction $h_{\boldsymbol{\Omega}}$ cancel to leave (4.22).

- (ii) The embedding of the Hofman-Maldacena magnon in $\mathbb{C}P^1$ is recovered as the $r \rightarrow 1$, $\boldsymbol{\Omega} \rightarrow \mathbf{0}$ limit of the new dyon.
- (iii) The embedding of the Hofman-Maldacena magnon in $\mathbb{R}P^2$ is recovered as the $r \rightarrow 1$, $|\boldsymbol{\Omega}| \rightarrow 1$ limit of the magnon in (i).
- (iv) The final type of solution is the embedding of Dorey's dyonic magnon in $\mathbb{R}P^3$. This solution turns out to be a composite of two of the elementary dyons of the form

$$(r, p, \boldsymbol{\Omega}^{(1)} - i\boldsymbol{\Omega}^{(2)}) + (1/r, p, \boldsymbol{\Omega}^{(1)} + i\boldsymbol{\Omega}^{(2)}), \quad (6.6)$$

where $\boldsymbol{\Omega}^{(i)}$, $i = 1, 2$ are two mutually orthogonal unit vectors, whose charge is

$$\Delta \mathcal{Q}_L = -2 \frac{1+r^2}{|r|} \left| \sin \frac{p}{2} \right| \Lambda - 2 \frac{1-r^2}{|r|} \left| \sin \frac{p}{2} \right| \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Omega}^{(1)} \boldsymbol{\Omega}^{(2)T} - \boldsymbol{\Omega}^{(2)} \boldsymbol{\Omega}^{(1)T} \end{pmatrix}. \quad (6.7)$$

Actually, the fact that Dorey's magnon is a composite object could have been guessed by considering the solitonic avatars. The new elementary dyon has a mass, eq. (5.10), $M_{\text{dyon}} = 2 \cos \alpha$, whereas the avatar of Dorey's magnon has a mass $M_{\text{Dorey}} = 4 \cos \alpha$ [12].

⁷Recall that $F \rightarrow -F$ under $\xi \rightarrow 1/\xi$.

From the point-of-view of the dressing transformation Dorey's dyonic magnon corresponds to a solution with 4 poles at $\lambda_i = (\xi, 1/\xi, 1/\xi^*, \xi^*)$ along with $\mu_i = (\xi^*, 1/\xi^*, 1/\xi, \xi)$. The first elementary dyon corresponds to the pair $\lambda_1 = \xi$ and $\lambda_2 = 1/\xi$ and the second to $\lambda_3 = 1/\xi^*$ and $\lambda_4 = \xi^*$ and the associated vectors ϖ_i , $i = 1, \dots, 4$, have the pair-wise constraints

$$\varpi_2 = \theta \varpi_1, \quad \varpi_4 = \theta \varpi_3. \quad (6.8)$$

However, because $\mu_1 = \lambda_4$ and $\mu_2 = \lambda_3^*$ there are the additional constraints that require

$$\varpi_1^\dagger \varpi_4 = 0, \quad \varpi_2^\dagger \varpi_3 = 0. \quad (6.9)$$

The individual elementary dyons therefore have the dressing data⁸

$$\varpi_1 = e_1 + ie_2 + \Omega_1, \quad \varpi_2 = -e_1 + ie_2 + \Omega_1, \quad (6.10)$$

and

$$\varpi_3 = \nu(-e_1 - ie_2 + \Omega_2), \quad \varpi_4 = \nu(e_1 - ie_2 + \Omega_2), \quad (6.11)$$

where we have used the overall-scaling symmetry to fix the scaling of $\varpi_{1,2}$. Then the additional constraints (6.9) require

$$\Omega_1^\dagger \cdot \Omega_2 = 0. \quad (6.12)$$

Dorey's dyon is then obtained as the special case when $\nu = 1$ and with the choice

$$\Omega_1 = \Omega^{(1)} - i\Omega^{(2)}, \quad \Omega_2 = \Omega^{(1)} + i\Omega^{(2)}, \quad (6.13)$$

for two orthogonal real vectors $\Omega^{(1,2)}$ with $|\Omega^{(1)}| = |\Omega^{(2)}|$. Notice this solves the constraint above. One can view setting $\nu = 1$ as fixing the relative position of the two elementary dyons in x and t . In addition, by choosing a suitable overall origin in x and t the two vectors $\Omega^{(1,2)}$ can be chosen to have unit length leading to the assignments in (6.6).

It is important to emphasize that in the context of CP^n , Dorey's dyon is a composite solution since ν can be thought of as determining the relative position of the two constituent dyons in x and t . However, the composite solution is also a solution of the sigma model with target space S^n . The dressing transformation in this case is identical up to the fact that one has to impose an additional reality condition on the solutions. This additional constraint leads to the conditions on the dressing data:

$$\varpi_4 = \varpi_1^*, \quad \varpi_3 = \varpi_2^*, \quad (6.14)$$

which enforces the choice $\nu = 1$ and (6.13). In other words, the additional reality constraint locks the two elementary dyons together to form Dorey's — now elementary — dyon. This is as it should be since the new elementary dyon is not by itself a solution in the S^n theory.

⁸As previously, Ω_1 and Ω_2 are orthogonal to e_1 and e_2 .

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A The generalized $\mathbb{C}P^2$ Dyon of [9]

In this appendix we shall recover the $\mathbb{C}P^2$ solution constructed by Abbott, Aniceto and Sax (AAS) in [9] from (4.32).

Consider (4.32) for $p = \pi$:

$$\begin{aligned} \mathbf{Z} = & \left[-\frac{1}{2} \left(2 |\omega_1|^2 e^{\frac{4rx}{r^2+1}} + \Omega^\dagger \Omega \right) - \frac{\Omega^\dagger \Omega}{r^2 - 1} \right] (\cos t \mathbf{e}_1 - \sin t \mathbf{e}_2) \\ & + |\omega_1|^2 e^{\frac{4rx}{r^2+1}} e^{-it} (\mathbf{e}_1 - i\mathbf{e}_2) - e^{\frac{2rx}{r^2+1}} e^{-i\frac{r^2-1}{r^2+1}t} \omega_1^* \Omega. \end{aligned} \quad (\text{A.1})$$

In order to simplify the comparison with the solution proposed in [9], we perform a $U(n+1)$ rotation

$$\mathbf{Z} \longrightarrow U\mathbf{Z}, \quad U = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \mathbf{0} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 1 \end{array} \right) \quad (\text{A.2})$$

so that

$$\begin{aligned} \tilde{\mathbf{Z}} = U\mathbf{Z} = & -e^{it} \frac{2(r^2-1)|\omega_1|^2 e^{\frac{4rx}{r^2+1}} + (r^2+1)\Omega^\dagger \Omega}{2\sqrt{2}(r^2-1)} \mathbf{e}_1 \\ & + e^{-it} \frac{2(r^2-1)|\omega_1|^2 e^{\frac{4rx}{r^2+1}} - (r^2+1)\Omega^\dagger \Omega}{2\sqrt{2}(r^2-1)} \mathbf{e}_2 \\ & - e^{-i\frac{r^2-1}{r^2+1}t} e^{\frac{2rx}{r^2+1}} \omega_1^* \Omega. \end{aligned} \quad (\text{A.3})$$

Then, we restrict ourselves to the case of $\mathbb{C}P^2$ and introduce the following parameterization (see [9], eq. 11)

$$\tilde{\mathbf{Z}} = N(t, x) \left(\sin \hat{\xi} \cos(\vartheta_2/2) e^{i\varphi_2/2} \mathbf{e}_1 + \sin \hat{\xi} \sin(\vartheta_2/2) e^{-i\varphi_2/2} \mathbf{e}_2 + \cos \hat{\xi} e^{i\varphi_1/2} \mathbf{e}_3 \right), \quad (\text{A.4})$$

together with $\Omega = |\Omega| \mathbf{e}_3$. Looking at the phases, this easily leads to

$$\varphi_2 = 2t \quad \text{and} \quad \varphi_1 = -2\omega t, \quad \text{with} \quad \omega = \frac{r^2-1}{r^2+1}, \quad (\text{A.5})$$

which is part of the ansatz used by AAS. In addition, they write

$$\cos \vartheta_2 = \text{sech} \left(\sqrt{1-\omega^2} 2x \right) = \text{sech} \left(\frac{4rx}{r^2+1} \right), \quad \hat{\xi} = \frac{\pi}{2} - e(x). \quad (\text{A.6})$$

Then, our solution matches theirs provided that

$$\tan\left(\frac{\vartheta_2}{2}\right) = -\frac{2(r^2 - 1)|\omega_1|^2 e^{\frac{4rx}{r^2+1}} - (r^2 + 1)\Omega^\dagger\Omega}{2(r^2 - 1)|\omega_1|^2 e^{\frac{4rx}{r^2+1}} + (r^2 + 1)\Omega^\dagger\Omega} = -\frac{e^{\frac{4rx}{r^2+1}} - 1}{e^{\frac{4rx}{r^2+1}} + 1}. \quad (\text{A.7})$$

This requires to fix

$$\frac{\Omega^\dagger\Omega}{|\omega_1|^2} = 2 \frac{r^2 - 1}{r^2 + 1} \equiv 2\omega \quad (\text{A.8})$$

which, taking (4.36) into account, amounts to fix the position of the magnon in space-time. Notice that this choice is only possible provided that $r^2 > 1$. Then,

$$\begin{aligned} \tilde{\mathbf{Z}} &= -e^{it} \frac{e^{\frac{4rx}{r^2+1}} + 1}{\sqrt{2}} |\omega_1|^2 \mathbf{e}_1 + e^{-it} \frac{e^{\frac{4rx}{r^2+1}} - 1}{\sqrt{2}} |\omega_1|^2 \mathbf{e}_2 - e^{-i\frac{r^2-1}{r^2+1}t} e^{\frac{2rx}{r^2+1}} \omega_1^* \Omega \\ \implies |N|^2 &= |\omega_1|^4 \left(e^{\frac{8rx}{r^2+1}} + 2 \frac{r^2 - 1}{r^2 + 1} e^{\frac{4rx}{r^2+1}} + 1 \right) \\ &\equiv 2|\omega_1|^4 e^{\sqrt{1-\omega^2} 2x} \cosh\left(\sqrt{1-\omega^2} 2x\right) \left[1 + \omega \operatorname{sech}\left(\sqrt{1-\omega^2} 2x\right) \right], \end{aligned} \quad (\text{A.9})$$

which finally leads to

$$\begin{aligned} \sin^2 \hat{\xi} = \cos^2 e(x) &= \frac{1}{|N|^2 \cos^2(\vartheta_2/2)} \left(\frac{e^{\frac{4rx}{r^2+1}} + 1}{\sqrt{2}} |\omega_1|^2 \right)^2 \\ &= \frac{1}{1 + \omega \operatorname{sech}\left(\sqrt{1-\omega^2} 2x\right)}, \end{aligned} \quad (\text{A.10})$$

which coincides precisely with eq. 13 of [9].

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